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Bicomplexes and integrable models

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Abstract. We associate bicomplexes with several integrable models in such a way that conserved currents are obtained by a simple iterative construction. Gauge transformations and dressings are discussed in this framework and several examples are presented, including the nonlinear Schrödinger and sine–Gordon equations, and some discrete models.

1. Introduction

Let $D = \mathrm{d} + A$ be the covariant exterior derivative associated with a connection 1-form A. The integrability condition of the linear equation $D\chi = 0$ for a vector valued function χ is the zero-curvature condition $F = \mathrm{d}A + A \wedge A = 0$ since $D^2\chi = F\,\chi$. In two dimensions where $A = -U\,\mathrm{d}x - V\,\mathrm{d}t$ with matrices U and V depending on coordinates x, t, the zero-curvature condition takes the form $U_t - V_x + [U, V] = 0$ (cf [1], for example) which can be rewritten in the form of a Lax equation. Soliton equations and integrable models are known to possess such a zero-curvature formulation with a connection (i.e. U and V) depending on a parameter, say λ (cf [1], for example).

This geometric formulation of integrable models is easily extended to generalized geometries, in particular in the sense of noncommutative geometry, where, on a basic level, the algebra of differential forms on a manifold is generalized to a differential calculus over an associative algebra $\mathcal{A}$ (for which the algebra of smooth functions on a manifold is an example) [2].

Recent work [3–5] shows that for many integrable models there is a zero-curvature formulation in which the linear system appears naturally in a form which depends *linearly* on the spectral parameter λ. However, translating the linear system into the form $\partial\chi/\partial x = U(x, t, \lambda)\,\chi$ and $\partial\chi/\partial t = V(x, t, \lambda)\,\chi$, as considered in [1], for example, one usually ends up with a *nonlinear* dependence of U and V on λ. An example in [1] for which U and V are linear in λ is the N-wave model (see p 309). A more important example where the connection depends linearly on λ is provided by the self-dual Yang–Mills equations from which many integrable models can be obtained by a reduction procedure [6]. In such cases we have $D = \delta - \lambda\,\mathsf{d}$ where δ and d are anticommuting linear maps satisfying $\mathsf{d}^2 = 0 = \delta^2$. They constitute what is known as a *bicomplex* (and need not satisfy the Leibniz rule as in the case of a bidifferential calculus). The linear system then reads

$$\delta\chi = \lambda\,\mathsf{d}\chi. \tag{1.1}$$

The most interesting point concerning this equation is not its simplicity in the dependence on λ, but rather the fact that expressing χ as a power series in λ leads in a very simple way to the conserved densities of the respective model. Moreover, behind this is a general iterative construction of δ-closed elements of a bicomplex. This is explained in more detail in section 2, which somewhat generalizes the framework of our previous papers. In particular, a modification of the above linear system by adding an inhomogeneous term is necessary, in general.

Applied to chiral models, the iterative construction of 'generalized conserved densities' in the sense of δ-closed elements of a bicomplex is precisely the construction of non-local conserved charges due to Brézin *et al* [7]. Our previous and the present work shows that the same method applies to most of the known soliton equations and integrable models (and perhaps to all of them). Surprisingly, in several cases the apparently *nonlocal* construction leads to *local* conserved currents and charges, rather than nonlocal ones.

In section 3 we apply gauge transformations and dressings to some (trivial) bicomplexes. Since we consider two separate 'generalized covariant derivatives' instead of one depending on a (spectral) parameter, a gauge transformation can be applied to just one of them. Such a dressing transformation deforms the bicomplex in a relatively simple way and the bicomplex conditions lead to equations for the transformation map. In this way one recovers several integrable models. Applying a gauge transformation simultaneously to both maps, d and δ, results in an equivalence transformation of the bicomplex, of course. We present several examples, including the nonlinear Schrödinger equation and the sine–Gordon equation and its discrete version, as well as a Toda field theory and a corresponding discretization. Furthermore, we briefly discuss generalizations of the self-dual Yang–Mills equations and reductions in the bicomplex framework. Section 4 contains some conclusions.

2. Weak bicomplexes and associated linear equation

Let $M = \bigoplus_{r\geqslant 0} M^r$ be an $\mathbb{N}_0$-graded linear space (over $\mathbb{R}$ or $\mathbb{C}$) and $\mathsf{d}, \delta : M^r \to M^{r+1}$, $\rho : M^r \to M^r$ linear maps satisfying

$$\rho\,\mathsf{d}^2 = 0 \qquad \delta^2 = 0 \qquad \delta\,\mathsf{d} + \rho\,\mathsf{d}\delta = 0. \tag{2.1}$$

Then $(M, \mathsf{d}, \delta, \rho)$ is called a *weak bicomplex*. If ρ is the identity map, then (M, d, δ) is a *bicomplex*. In terms of $\mathsf{d}_\lambda = \delta - \lambda\,\mathsf{d}$ with a constant λ, the three bicomplex equations can then be combined into the single condition $\mathsf{d}_\lambda^2 = 0$ (for all λ).

We are interested in the case where the weak bicomplex maps depend on certain objects (e.g. functions or operators) in such a way that the above bicomplex equations are satisfied if these objects are solutions of some (e.g. differential or operator) equations. Of particular interest are those cases where the bicomplex conditions become equivalent to a certain (e.g. nonlinear partial differential) equation.

We assume that, for some $s \in \mathbb{N}$, there is a (nonvanishing) $\chi^{(0)} \in M^{s-1}$ with

$$\rho\,\mathsf{d}J^{(0)} = 0 \qquad \text{where} \quad J^{(0)} = \delta\chi^{(0)}. \tag{2.2}$$

Let us define

$$J^{(1)} = \mathsf{d}\chi^{(0)}. \tag{2.3}$$

Then

$$\delta J^{(1)} = -\rho\,\mathsf{d}\delta\,\chi^{(0)} = 0. \tag{2.4}$$

If the δ-closed element $J^{(1)}$ is δ-exact, then

$$J^{(1)} = \delta\chi^{(1)} \tag{2.5}$$

$$\begin{tikzcd}
M^{s-1} \arrow[dd] & & \chi^{(0)} \arrow[dl] \arrow[dr, "d"] & & \chi^{(1)} \arrow[dl, "\delta"'] \arrow[dr, "d"] & & \\
M^{s} \arrow[dd] & J^{(0)} \arrow[dr, "\rho d"] & & J^{(1)} \arrow[dl, "\delta"'] \arrow[dr, "\rho d"] & & J^{(2)} \arrow[dl, "\delta"'] & \ldots \\
M^{s+1} & & 0 & & 0 & &
\end{tikzcd}$$

Figure 1. The iterative construction of δ-closed elements $J^{(m)} \in M^s$.

with some $\chi^{(1)} \in M^{s-1}$. Next we define

$$J^{(2)} = \mathsf{d}\chi^{(1)}. \tag{2.6}$$

Then

$$\delta J^{(2)} = -\rho\, \mathsf{d}\delta\, \chi^{(1)} = -\rho\, \mathsf{d}J^{(1)} = -\rho\, \mathsf{d}^2\chi^{(0)} = 0. \tag{2.7}$$

If the δ-closed element $J^{(2)}$ is δ-exact, then

$$J^{(2)} = \delta\chi^{(2)} \tag{2.8}$$

with some $\chi^{(2)} \in M^{s-1}$. This can be iterated further and leads to a (possibly infinite) chain (see figure 1) of elements $J^{(m)}$ of M^s and $\chi^{(m)} \in M^{s-1}$ satisfying

$$J^{(m+1)} = \mathsf{d}\chi^{(m)} = \delta\chi^{(m+1)}. \tag{2.9}$$

More precisely, the above iteration continues from the mth to the $(m+1)$th level as long as $\delta J^{(m)} = 0$ implies $J^{(m)} = \delta\chi^{(m)}$ with an element $\chi^{(m)} \in M^{s-1}$. Of course, there is no obstruction to the iteration if $H^s_\delta(M)$ is trivial, i.e. when all δ-closed elements of M^s are δ-exact. In general, the latter condition is too strong, however, though in several examples it can be easily verified [3].

Introducing

$$\chi = \sum_{m\geqslant 0} \lambda^m\, \chi^{(m)} \tag{2.10}$$

with a parameter λ, the essential ingredients of the above iteration procedure are summarized in

$$\delta(\chi - \chi^{(0)}) = \lambda\, \mathsf{d}\chi \tag{2.11}$$

which we call the *linear equation* associated with the bicomplex†.

Applying δ to the bicomplex linear equation and using (2.1), we find

$$\rho\, \mathsf{d}\delta\, \chi = 0. \tag{2.12}$$

Hence χ has to satisfy the same condition as we put on the initial data $\chi^{(0)}$. We may thus think of the above iteration as a discrete process in the space of solutions of this equation.

† Independent of the choice of ρ, we are led to the same form of the linear equation (2.11). The map ρ enters via the initial condition $\rho\, \mathsf{d}\delta\, \chi^{(0)} = 0$. The linear equation then implies $[\delta^2 - \lambda\,(\delta\, \mathsf{d} + \rho\, \mathsf{d}\delta) + \lambda^2 \rho\, \mathsf{d}^2]\chi = (\delta^2 - \lambda\, \rho\, \mathsf{d}\delta)\chi^{(0)} = 0$.

Now we can turn things around. Given a bicomplex, we may start with the linear equation (2.11). Let us suppose that it admits a (non-trivial) solution χ as a (formal) power series in the parameter λ:

$$\chi = \sum_{m=0}^{N} \lambda^m \chi^{(m)} \tag{2.13}$$

with $N \in \mathbb{N} \cup \{\infty\}$. The linear equation leads to

$$\delta \chi^{(m)} = \mathsf{d}\chi^{(m-1)} \qquad m = 1, \ldots, N \qquad \mathsf{d}\chi^{(N)} = 0 \tag{2.14}$$

where the last equation has to be dropped if $N = \infty$. As a consequence, the $J^{(m+1)} = \mathsf{d}\chi^{(m)}$ ($m = 0, \ldots, N-1$) are δ-exact. Even if the cohomology $H^s_\delta(M)$ is *not* trivial, the solvability of the linear equation ensures that the δ-closed $J^{(m)}$ appearing in the iteration are δ-exact†. This observation somewhat generalizes the framework of our previous papers and indeed appears to be necessary in order to cover examples such as the nonlinear Schrödinger equation (see the following section).

A priori, the mathematics presented above has little to do with conservation laws. However, formulating integrable models such as KdV, KP, chiral models and the like in the bicomplex framework has demonstrated that the δ-exact $J^{(m)}$ (where $s = 1$) are directly or somewhat indirectly related to the known conserved densities of the respective models [3–5]. This is also confirmed by the examples treated in the following section.

The features usually attributed to *soliton equations* demand a high level of order and predictability, in complete contrast with chaotic systems, which, in this sense, form the dark side of nonlinear dynamics. Soliton systems were found to possess an infinite set of conservation laws. This was taken as a (partial) explanation for the high order of simplicity of their scattering behaviour. If there is an infinite chain of independent δ-exact elements in a bicomplex associated with some (integro-differential, difference, operator) equation, this is certainly also a property expressing a high degree of order. In this sense the above structure should also be of interest beyond the context of integrable models.

The freedom which enters the formalism through the possible choice of a map ρ different from the identity and also the possibility of considering $s > 1$ will not be explored in this work. Hence, in the following we restrict our considerations to the simpler structure of a bicomplex and the case where $s = 1$.

In the examples which we present in the following sections, the bicomplex space is always chosen as $M = M^0 \otimes \Lambda_n$ where $\Lambda_n = \bigoplus_{r=0}^{n} \Lambda^r$ is the exterior algebra of a (complex) n-dimensional vector space with a basis ξ^r, $r = 1, \ldots, n$, of Λ^1. It is then sufficient to define the bicomplex maps d and δ on M^0 since via

$$\mathsf{d}\Bigg(\sum_{i_1,\ldots,i_r=1}^{n} \phi_{i_1\ldots i_r}\, \xi^{i_1} \cdots \xi^{i_r} \Bigg) = \sum_{i_1,\ldots,i_r=1}^{n} (\mathsf{d}\phi_{i_1\ldots i_r})\, \xi^{i_1} \cdots \xi^{i_r} \tag{2.15}$$

(and correspondingly for δ) they extend as linear maps to the whole of M. In the case of Λ_2 we denote the two basis elements of Λ^1 as τ, ξ.

3. Gauge transformations and dressings of bicomplexes

Let (M, d, δ) be a bicomplex. A *gauge transformation* is a map of this bicomplex to another bicomplex $(M, \mathsf{d}', \delta')$ induced by an isomorphism g of M such that

$$\mathsf{d}'\phi = g^{-1}\mathsf{d}(g\phi) \qquad \delta'\phi = g^{-1}\delta(g\phi). \tag{3.1}$$

† If the cohomology condition holds, then the above iterative procedure provides us with a solution of the linear equation. Otherwise we have to show that the linear equation has a sufficiently nontrivial solution.

Indeed, it is easily verified that d' and δ' satisfy the bicomplex conditions (2.1) (with $\rho = \mathrm{id}$) if d and δ satisfy them.

There are at least two simple ways to deform a bicomplex such that two of the bicomplex conditions (2.1) remain satisfied. In both cases we leave one of the two bicomplex maps, say δ, unchanged. We call transformations of this kind *dressing transformations*.

The first way is to transform d to

$$\tilde{\mathsf{D}}\phi = \mathsf{d}\phi + [\delta, v]\phi = \mathsf{d}\phi + \delta(v\phi) - v\,\delta\phi \tag{3.2}$$

where v is a linear map $M \to M$. Then

$$\tilde{\mathsf{D}}\delta + \delta\tilde{\mathsf{D}} = \mathsf{d}\delta + \delta\mathsf{d} = 0 \tag{3.3}$$

using $\delta^2 = 0$, so that all bicomplex equations besides $\tilde{\mathsf{D}}^2 = 0$ are identically satisfied. The remaining condition takes the form

$$\mathsf{d}\delta\,(v\phi) - \mathsf{d}(v\delta\phi) - \delta(v\delta\phi) + \delta(v\,\mathsf{d}\phi) - v\delta\,\mathsf{d}\phi + v\delta(v\delta\phi) = 0. \tag{3.4}$$

The problem is now to find d and δ such that the last equation reduces to an interesting equation for v independent of ϕ.

Example (KP equation). Let $M = C^\infty(\mathbb{R}^3) \otimes \Lambda_2$. In terms of coordinates t, x, y on $\mathbb{R}^3$ we define bicomplex maps d and δ via

$$\mathsf{d}f = (f_t - f_{xxx})\,\tau + \tfrac{1}{2}(f_y - f_{xx})\,\xi \qquad \delta f = \tfrac{3}{2}(f_y + f_{xx})\,\tau + f_x\,\xi \tag{3.5}$$

for $f \in C^\infty(\mathbb{R}^3) = M^0$. The bicomplex equations (2.1) are then identically satisfied. Deforming d to

$$\begin{aligned}\tilde{\mathsf{D}}f &= \mathsf{d}f + \delta(vf) - v\delta f \\ &= [f_t - f_{xxx} + \tfrac{3}{2}(v_y + v_{xx})f + 3v_x f_x]\,\tau + \tfrac{1}{2}(f_y - f_{xx} + 2v_x f)\,\xi\end{aligned} \tag{3.6}$$

with $v \in M^0$ (which, by multiplication, acts linearly on M), $\tilde{\mathsf{D}}^2 = 0$ becomes

$$v_{xt} - \tfrac{1}{4}v_{xxxx} + 3v_x v_{xx} - \tfrac{3}{4}v_{yy} = 0 \tag{3.7}$$

which is equivalent to the KP equation for the field $u = -v_x$.

The second kind of dressing transformation is to transform d to

$$\mathsf{D}\phi = G^{-1}\mathsf{d}(G\phi) \tag{3.8}$$

where G is an isomorphism of M. Then $\mathsf{D}^2\phi = G^{-1}\mathsf{d}^2(G\phi) = 0$ so that all bicomplex equations besides $\delta\mathsf{D} + \mathsf{D}\delta = 0$ are identically satisfied. The remaining condition results in the following equation involving G:

$$\delta[G^{-1}\,\mathsf{d}(G\phi)] + G^{-1}\,\mathsf{d}(G\,\delta\phi) = 0. \tag{3.9}$$

Again, the game is to find d and δ such that this reduces to an interesting equation for G independent of ϕ. The following subsections present several examples.

3.1. A unifying bicomplex framework for some integrable models

Let M^0 be the space of 2×2 matrices with entries in $C^\infty(\mathbb{R}^3)$ and $M = M^0 \otimes \Lambda_2$. In terms of coordinates t, x, y on $\mathbb{R}^3$ we define linear maps $\mathrm{d}, \delta : M^0 \to M^1$ via

$$\mathrm{d}\phi = \phi_t\,\tau + \phi_x\,\xi \qquad \delta\phi = \phi_y\,\tau + \frac{1}{2\mathrm{i}}\,(I - \sigma_3)\phi\,\xi \tag{3.10}$$

where I is the 2×2 unit matrix and $\sigma_3 = \mathrm{diag}(1, -1)$. The bicomplex conditions are then identically satisfied. The δ-cohomology is not trivial. For example, elements of the form $(c(t, x), 0)\,\xi$ are δ-closed but not δ-exact.

Now we dress d with some invertible 2×2 matrix G as follows:

$$\mathsf{D}\phi = G^{-1}\mathsf{d}(G\phi) = (\phi_t - V\,\phi)\,\tau + (\phi_x - U\,\phi)\,\xi \tag{3.11}$$

where

$$U = -G^{-1}G_x \qquad V = -G^{-1}G_t. \tag{3.12}$$

In terms of U and V, the bicomplex equation $\mathsf{D}^2 = 0$ reads

$$U_t - V_x + [U, V] = 0 \tag{3.13}$$

which is an identity in the case under consideration. The only nontrivial bicomplex equation is $\delta\mathsf{D} + \mathsf{D}\delta = 0$, which takes the form

$$U_y - \frac{\mathrm{i}}{2}\,[\sigma_3, V] = 0. \tag{3.14}$$

Example. Let

$$G = \exp\left(\frac{\mathrm{i}}{2}\,(I - \sigma_3)\,t\right)\exp\left(\frac{\mathrm{i}}{2}\,\sigma_2\,u\right) \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{3.15}$$

where u does not depend on t. Then

$$U = \begin{pmatrix} 0 & -u_x/2 \\ u_x/2 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} -\sin^2(u/2) & \sin(u/2)\,\cos(u/2) \\ \sin(u/2)\,\cos(u/2) & -\cos^2(u/2) \end{pmatrix} \tag{3.16}$$

and (3.14) is equivalent to the sine–Gordon equation $u_{xy} = \sin u$.

Now we decompose V as follows:

$$V = \mathrm{i}\,(V^+ + V^-)\,\sigma_3 \tag{3.17}$$

where

$$\sigma_3\,V^+\,\sigma_3 = V^+ \qquad \sigma_3\,V^-\,\sigma_3 = -V^-. \tag{3.18}$$

Then (3.14) becomes

$$V^- = U_y. \tag{3.19}$$

This implies $\sigma_3\,U_y\,\sigma_3 = -U_y$, which restricts U to the following form:

$$U = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{3.20}$$

with functions q and r, up to addition of terms on the diagonal which do not depend on y. If the latter vanish, then

$$\sigma_3\,U\,\sigma_3 = -U. \tag{3.21}$$

Using (3.19) to eliminate V^- from (3.13), we obtain the two equations

$$V^+ = \partial_x^{-1}(U^2)_y \tag{3.22}$$

and

$$\mathrm{i}\,U_t = -[U_{xy} - 2\,U\,\partial_x^{-1}(U^2)_y]\,\sigma_3. \tag{3.23}$$

Here ∂_x^{-1} means integration with respect to x. In conclusion, we note the following. Let U be of the form (3.20). Defining V via (3.17), (3.19) and (3.22), then the bicomplex equations are satisfied iff (3.23) holds. Equation (3.23) generalizes the nonlinear Schrödinger equation. Indeed, setting $y = x$ (which reduces the system to two dimensions) and choosing $q = \bar{\psi}$ and $r = \psi$ with a complex function ψ with complex conjugate $\bar{\psi}$, the equation (3.23) becomes equivalent to $\mathrm{i}\,\psi_t = -\psi_{xx} + 2\,|\psi|^2\,\psi$ (and its complex conjugate). Relations with the AKNS formalism are rather obvious (cf [8], for example).

Suppose we have an invertible matrix, say $\tilde{\chi}$, the entries of which are functions of two coordinates, say t and x. Then the identity

$$(\tilde{\chi}_x \tilde{\chi}^{-1})_t = (\tilde{\chi}_t \tilde{\chi}^{-1})_x + [\tilde{\chi}_t \tilde{\chi}^{-1}, \tilde{\chi}_x \tilde{\chi}^{-1}] \tag{3.24}$$

implies that

$$\mathrm{tr}(\tilde{\chi}_x \tilde{\chi}^{-1})_t = \mathrm{tr}(\tilde{\chi}_t \tilde{\chi}^{-1})_x \tag{3.25}$$

which has the form of a conservation law. However, the nontrivial task is to ensure that the brackets on both sides contain only terms which are *local*† in the field we are interested in (see also [9]).

Let us now turn to the associated linear system $\delta\chi = \lambda\,\mathsf{D}\chi$ for a matrix $\chi \in M^0$, i.e.

$$\chi_y = \lambda\,(\chi_t - V\,\chi) \qquad e_-\chi = \mathrm{i}\,\lambda\,(\chi_x - U\,\chi) \tag{3.26}$$

where $e_\pm = (I \pm \sigma_3)/2$. The second equation implies

$$e_+\chi_x = e_+ U\chi \tag{3.27}$$

and, using (3.21),

$$e_+\chi_x = U\,e_-\chi = \mathrm{i}\lambda(U\,\chi_x - U^2\chi). \tag{3.28}$$

On the other hand, differentiating the second equation of the linear system with respect to x leads to

$$e_-\chi_x = \mathrm{i}\,\lambda\,(\chi_{xx} - U_x\,\chi - U\,\chi_x) \tag{3.29}$$

and combining the last two equations gives

$$\chi_x = \mathrm{i}\,\lambda\,(\chi_{xx} - (U_x + U^2)\,\chi). \tag{3.30}$$

Assuming that U is invertible (i.e. $qr \neq 0$), (3.27) can also be written as

$$e_-\chi = e_- U^{-1}\chi_x. \tag{3.31}$$

The linear system implies $e_-\chi^{(0)} = 0$, which is solved by $\chi^{(0)} = e_+$. In particular, it follows that χ is not invertible (as a formal power series in λ). Let us consider instead

$$\tilde{\chi} = \chi + e_- \tag{3.32}$$

which is invertible. Using

$$\chi = e_+\chi + e_-\chi = e_+\tilde{\chi} + e_- U^{-1}\tilde{\chi}_x \tag{3.33}$$

(3.30) becomes

$$\tilde{\chi}_x = \mathrm{i}\,\lambda\,(\tilde{\chi}_{xx} - (e_- U + e_+ U_x U^{-1})\,\tilde{\chi}_x - (e_- U_x + e_+ U^2)\,\tilde{\chi}). \tag{3.34}$$

Introducing θ via

$$\tilde{\chi}_x \tilde{\chi}^{-1} = \lambda\,\theta \tag{3.35}$$

the last equation takes the form

$$\theta = -\mathrm{i}\,(e_- U_x + e_+ U^2) + \mathrm{i}\,\lambda\,(\theta_x - (e_- U + e_+ U_x U^{-1})\,\theta) + \mathrm{i}\,\lambda^2\,\theta^2. \tag{3.36}$$

Inserting the power series expansion

$$\theta = \sum_{k \geqslant 0} \lambda^k\,\theta^{(k)} \tag{3.37}$$

† In the sense of not involving integrals of the field.

in (3.36), we find

$$\theta^{(0)} = -\mathrm{i}\,(e_- U_x + e_+ U^2) \qquad \theta^{(1)} = e_+ U U_x + e_-(U_{xx} - U^3) \tag{3.38}$$

and

$$\theta^{(k)} = \mathrm{i}\,(\theta^{(k-1)}_x - (e_- U + e_+ U_x U^{-1})\,\theta^{(k-1)}) + \mathrm{i}\sum_{j=0}^{k-2} \theta^{(j)}\,\theta^{(k-2-j)} \tag{3.39}$$

for $k > 1$. According to the general argument given above, the quantities

$$w^{(k)} = \mathrm{tr}\;\theta^{(k)} \tag{3.40}$$

are conserved in a two-dimensional sense (with respect to both coordinate pairs t, x and y, x). A more explicit form of the conservation laws is given in the addendum below. We find

$$w^{(0)} = -\mathrm{i}\;\mathrm{tr}(e_+ U^2) = -\mathrm{i}\,q\,r \qquad w^{(1)} = \mathrm{tr}(e_+ U U_x) = q\,r_x \tag{3.41}$$

and

$$w^{(k)} = \mathrm{i}\left(w^{(k-1)}_x - \frac{q_x}{q}\,w^{(k-1)}\right) + \mathrm{i}\sum_{j=0}^{k-2} w^{(j)}\,w^{(k-2-j)}. \tag{3.42}$$

With $y = x$ and $q = \bar{\psi}$, $r = \psi$, one recovers from the last equations the conserved densities of the nonlinear Schrödinger equation. Choosing $r = -q = u_x/2$ where $u = u(x, y)$, the $w^{(k)}$ reproduce the conserved densities of the sine–Gordon equation.

Addendum. The first equation of the linear system can be rewritten as follows:

$$\tilde{\chi}_y \tilde{\chi}^{-1} = \lambda\,(\tilde{\chi}_t \tilde{\chi}^{-1} - V\,(e_+ + \lambda\,e_- U^{-1}\theta)). \tag{3.43}$$

With its help we obtain

$$\begin{aligned}\theta_y &= \frac{1}{\lambda}(\tilde{\chi}_x \tilde{\chi}^{-1})_y = \frac{1}{\lambda}(\tilde{\chi}_y \tilde{\chi}^{-1})_x + [\tilde{\chi}_y \tilde{\chi}^{-1}, \theta] \\ &= (\tilde{\chi}_t \tilde{\chi}^{-1} - V\,(e_+ + \lambda\,e_- U^{-1}\theta))_x + \lambda[\tilde{\chi}_t \tilde{\chi}^{-1} - V\,(e_+ + \lambda\,e_- U^{-1}\theta), \theta].\end{aligned} \tag{3.44}$$

Similarly we obtain

$$\begin{aligned}\theta_t &= \frac{1}{\lambda}(\tilde{\chi}_x \tilde{\chi}^{-1})_t = \frac{1}{\lambda}(\tilde{\chi}_t \tilde{\chi}^{-1})_x + [\tilde{\chi}_t \tilde{\chi}^{-1}, \theta] \\ &= (\lambda^{-2}\tilde{\chi}_y \tilde{\chi}^{-1} + \lambda^{-1} V\,(e_+ + \lambda\,e_- U^{-1}\theta))_x + [\lambda^{-1}\tilde{\chi}_y \tilde{\chi}^{-1} + V\,(e_+ + \lambda\,e_- U^{-1}\theta), \theta].\end{aligned} \tag{3.45}$$

It follows that $w = \mathrm{tr}\;\theta$ satisfies the conservation laws

$$w_y = \mathrm{tr}(\tilde{\chi}_t \tilde{\chi}^{-1} - V\,(e_+ + \lambda\,e_- U^{-1}\theta))_x \tag{3.46}$$

$$w_t = \mathrm{tr}(\lambda^{-2}\tilde{\chi}_y \tilde{\chi}^{-1} + \lambda^{-1} V\,(e_+ + \lambda\,e_- U^{-1}\theta))_x. \tag{3.47}$$

If we apply a gauge transformation with $g = G^{-1}$ to the bicomplex associated with the nonlinear Schrödinger equation (where $y = x$), we obtain

$$\mathrm{D}'\phi = \phi_t\,\tau + \phi_x\,\xi \tag{3.48}$$

$$\delta'\phi = G\,\delta(G^{-1}\phi) = (\phi_x - G_x G^{-1}\phi)\,\tau + \frac{1}{2\mathrm{i}}\,(I - S)\phi\,\xi \tag{3.49}$$

where $S = G\sigma_3 G^{-1}$. The bicomplex conditions now take the form

$$S_x = [G_x G^{-1}, S] \qquad S_t = 2\,\mathrm{i}\,(G_x G^{-1})_x. \tag{3.50}$$

The first equation leads to $G_x G^{-1} = -(1/2) S S_x$ using $G^{-1} G_x \, \sigma_3 + \sigma_3 \, G^{-1} G_x = 0$ (cf (3.21)), and the second takes the form

$$S_t = -\mathrm{i}\,(S S_x)_x = -\mathrm{i}\,\left(S S_x - \frac{1}{2}(S S_x + S_x S) \right)_x = -\frac{\mathrm{i}}{2}\,[S, S_{xx}] \tag{3.51}$$

where we used $S^2 = I$. This is the Heisenberg magnet equation

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} \tag{3.52}$$

where we have set $S = \vec{S} \cdot \vec{\sigma}$. We have thus reproduced the equivalence of the nonlinear Schrödinger equation and the Heisenberg magnet (cf [1], for example).

3.2. *Further bicomplex formulations of integrable models*

3.2.1. Sine–Gordon equation again . For $z : \mathbb{R}^2 \to \mathbb{C}$ we define

$$\mathsf{d}z = \tfrac{1}{2}(\bar{z} - z)\,\tau + z_x\,\xi \tag{3.53}$$

$$\delta z = z_t\,\tau + \tfrac{1}{2}(\bar{z} - z)\,\xi \tag{3.54}$$

where $\bar{z}$ denotes the complex conjugate of z. Then $(M = C^\infty(\mathbb{R}^2, \mathbb{C}) \otimes \Lambda_2, \mathsf{d}, \delta)$ is a (trivial) bicomplex. Deforming d to

$$\mathsf{D}z = \mathrm{e}^{-\mathrm{i}\varphi/2}\,\mathsf{d}(\mathrm{e}^{\mathrm{i}\varphi/2} z) = \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\varphi}\bar{z} - z)\,\tau + \left(z_x + \frac{\mathrm{i}}{2}\varphi_x\,z \right)\,\xi \tag{3.55}$$

$\delta\mathsf{D} + \mathsf{D}\delta = 0$ turns out to be equivalent to the sine–Gordon equation $\varphi_{tx} = \sin\varphi$. The first cohomology group of δ is not trivial. In particular, ξ is δ-closed, but not in $\delta(M^0)$.

With $\chi^{(0)} = 1$ the linear equation $\delta\chi = \lambda\,\mathsf{D}\chi$ consists of the two equations

$$\chi_t = \frac{\lambda}{2}(\mathrm{e}^{-\mathrm{i}\varphi}\,\bar{\chi} - \chi) \qquad \frac{1}{2}(\bar{\chi} - \chi) = \lambda\,\left(\chi_x + \frac{\mathrm{i}}{2}\varphi_x\,\chi \right). \tag{3.56}$$

Writing $\chi = \alpha + \mathrm{i}\,\beta$, the second equation of the linear system becomes

$$\alpha_x - \tfrac{1}{2}\,\varphi_x\,\beta = 0 \qquad \beta = -\lambda\,(\beta_x + \tfrac{1}{2}\,\varphi_x\,\alpha). \tag{3.57}$$

Eliminating β and setting $\alpha = \mathrm{e}^{-\lambda\gamma}$ with a function γ yields

$$\gamma_x = \frac{1}{4}{\varphi_x}^2 - \lambda\,\left(\gamma_{xx} - \frac{\varphi_{xx}}{\varphi_x}\gamma_x \right) + \lambda^2\,{\gamma_x}^2. \tag{3.58}$$

From the first equation of the linear system we obtain

$$\alpha_t = \frac{\lambda}{2}(\alpha\cos\varphi - \beta\sin\varphi - \alpha). \tag{3.59}$$

After some simple manipulations one arrives at the conservation law

$$(\gamma_x)_t = -\left(\lambda\,\frac{\sin\varphi}{\varphi_x}\,\gamma_x + \frac{1}{2}\cos\varphi \right)_x . \tag{3.60}$$

Inserting the power series expansion for γ with respect to λ, one obtains the conserved quantities

$$\gamma_x^{(0)} = \tfrac{1}{4}{\varphi_x}^2 \qquad \gamma_x^{(1)} = -\tfrac{1}{8}({\varphi_x}^2)_x \qquad \ldots \tag{3.61}$$

of the sine–Gordon equation. Similar calculations can be performed in the case of the following models.

3.2.2. Discrete sine–Gordon equation. Let M^0 be the space of complex functions on an infinite plane square lattice. We define linear maps $\mathsf{d}, \delta : M^0 \to M^1$ by

$$(\delta z)_{\mathrm{S}} = (z_{\mathrm{E}} - z_{\mathrm{S}})\,\tau + a\,(\bar{z}_{\mathrm{W}} - z_{\mathrm{S}})\,\xi \tag{3.62}$$

$$(\mathsf{d}z)_{\mathrm{S}} = a\,(\bar{z}_{\mathrm{E}} - z_{\mathrm{S}})\,\tau + (z_{\mathrm{W}} - z_{\mathrm{S}})\,\xi \tag{3.63}$$

where a subscript N, S, E or W means taking the value at the north, south, east and west points, respectively, of a common elementary square of the lattice (cf [11] for this notation). It is easily verified that $(M = M^0 \otimes \Lambda_2, \mathsf{d}, \delta)$ is a bicomplex. Now we deform d to

$$(\mathsf{D}z)_{\mathrm{S}} = \mathrm{e}^{-\mathrm{i}\varphi_{\mathrm{S}}/2}(\mathsf{d}(\mathrm{e}^{\mathrm{i}\varphi/2}z))_{\mathrm{S}} = a(\mathrm{e}^{-\mathrm{i}(\varphi_{\mathrm{S}}+\varphi_{\mathrm{E}})/2}\bar{z}_{\mathrm{E}} - z_{\mathrm{S}})\,\tau + (\mathrm{e}^{-\mathrm{i}(\varphi_{\mathrm{S}}-\varphi_{\mathrm{W}})/2}z_{\mathrm{W}} - z_{\mathrm{S}})\,\xi \tag{3.64}$$

with a real function φ. Then $\delta\mathsf{D} + \mathsf{D}\delta = 0$ is equivalent to

$$\mathrm{e}^{\mathrm{i}(\varphi_{\mathrm{N}}-\varphi_{\mathrm{E}})/2} - \mathrm{e}^{-\mathrm{i}(\varphi_{\mathrm{S}}-\varphi_{\mathrm{W}})/2} = a^2(\mathrm{e}^{\mathrm{i}(\varphi_{\mathrm{N}}+\varphi_{\mathrm{W}})/2} - \mathrm{e}^{-\mathrm{i}(\varphi_{\mathrm{S}}+\varphi_{\mathrm{E}})/2}). \tag{3.65}$$

Taking real and imaginary parts and using some trigonometry, one proves that this is equivalent to

$$\sin[\tfrac{1}{4}(\varphi_{\mathrm{N}} + \varphi_{\mathrm{S}} - \varphi_{\mathrm{E}} - \varphi_{\mathrm{W}})] = a^2 \sin[\tfrac{1}{4}(\varphi_{\mathrm{N}} + \varphi_{\mathrm{S}} + \varphi_{\mathrm{E}} + \varphi_{\mathrm{W}})] \tag{3.66}$$

which is the discrete sine–Gordon equation [10].

3.2.3. Toda field theory. Let M^0 be the algebra of functions on $\mathbb{R}^2 \times \mathbb{Z}$ which are smooth in the first two arguments. We write $f_k(t, x) = f(t, x, k)$ for $k \in \mathbb{Z}$. Supplied with the linear maps defined by

$$\delta f_k = (\partial_t f_k)\,\tau + (f_{k+1} - f_k)\,\xi \qquad \mathsf{d}f_k = -(f_k - f_{k-1})\,\tau + (\partial_x f_k)\,\xi \tag{3.67}$$

$M = M^0 \otimes \Lambda_2$ becomes a bicomplex. The bicomplex conditions (2.1) are identically satisfied. Now we dress d as follows:

$$\mathsf{D}f_k = \mathrm{e}^{-q_k}\,\mathsf{d}(\mathrm{e}^{q_k} f_k) = (\mathrm{e}^{q_{k-1}-q_k} f_{k-1} - f_k)\,\tau + (\partial_x f_k + (\partial_x q_k) f_k)\,\xi \tag{3.68}$$

where $q \in M^0$. Then $\delta\mathsf{D} + \mathsf{D}\delta = 0$ is equivalent to the Toda field equation

$$\partial_t \partial_x q_k = \mathrm{e}^{q_k - q_{k+1}} - \mathrm{e}^{q_{k-1}-q_k}. \tag{3.69}$$

See also [3, 5, 12] for some related work.

3.2.4. A generalization of Hirota's difference equation. Hirota's difference equation is a discretization of the Toda field theory [13]. Let M^0 be the algebra of functions of n discrete variables $x^1, \ldots, x^n$ and $M = M^0 \otimes \Lambda_n$. M becomes a bicomplex with d and δ determined by

$$\mathsf{d}f = \sum_i a_i\,(RS_i f - f)\,\xi^i \qquad \delta f = \sum_i b_i\,(S_i f - f)\,\xi^i \tag{3.70}$$

where $(S_i f)(x^1, \ldots, x^n) = f(x^1, \ldots, x^{i-1}, x^i + 1, x^{i+1}, \ldots, x^n)$. R is an automorphism of the algebra of functions commuting with S_i, and a_i, b_i are constants. The bicomplex equations are then identically satisfied. Now we deform d to

$$\mathsf{D}f = \mathrm{e}^{-q}\,\mathsf{d}(\mathrm{e}^q f) = \sum_i a_i\,(\mathrm{e}^{RS_i q - q}\,RS_i f - f)\,\xi^i \tag{3.71}$$

with a function q of the discrete variables. Then $\delta\mathsf{D} + \mathsf{D}\delta = 0$ yields

$$a_i b_j\,(\mathrm{e}^{RS_i S_j q - S_j q} - \mathrm{e}^{RS_i q - q}) = a_j b_i\,(\mathrm{e}^{RS_i S_j q - S_i q} - \mathrm{e}^{RS_j q - q}). \tag{3.72}$$

3.3. Remarks on self-dual Yang–Mills equations and reductions

Let $\mathcal{A}$ be an associative (and not necessarily commutative) algebra over $\mathbb{R}$ or $\mathbb{C}$. Furthermore, let $M = \Omega \otimes_{\mathcal{A}} \mathcal{A}^m$ where $(\Omega, \mathsf{d}, \delta)$ is a bidifferential calculus over $\mathcal{A}$ (cf [3]), so that d and δ satisfy the (graded) Leibniz rule. If we apply dressings of the second kind to *both* generalized exterior derivatives, we obtain a bicomplex $(M, \mathsf{D}, \mathcal{D})$ where

$$\mathsf{D}\phi = \mathsf{d}\phi + A\,\phi \qquad \mathcal{D}\phi = \delta\phi + B\,\phi \tag{3.73}$$

if the bicomplex conditions

$$\mathsf{d}A + A^2 = 0 \qquad \delta B + B^2 = 0 \qquad \mathsf{d}B + \delta A + AB + BA = 0 \tag{3.74}$$

are satisfied. The first two equations are solved, of course, with $A = G^{-1}\,\mathsf{d}G$ and an analogous expression for B. Applying suitable gauge transformations, we obtain equivalent bicomplexes (M, D', δ) and $(M, \mathsf{d}, \mathcal{D}')$, in particular.

Let us now specialize $\mathcal{A}$ to the commutative algebra of smooth functions of $2n$ variables $x^i, y^i, i = 1, \ldots, n$, and set

$$\mathsf{d}f = \sum_i \frac{\partial f}{\partial x^i}\,\xi^i \qquad \delta f = \sum_i \frac{\partial f}{\partial y^i}\,\xi^i \tag{3.75}$$

where ξ^i, $i = 1, \ldots, n$, is a basis of Λ_n. Since B can be transformed to zero by a gauge transformation, it is sufficient to consider (M, D, δ), which is a bicomplex iff

$$\mathsf{d}A + A^2 = 0 \qquad \delta A = 0. \tag{3.76}$$

For $n = 2$ this is gauge equivalent to the self-dual Yang–Mills equations [3, 16]. Higher-dimensional generalizations of the above kind with $n > 2$ have been considered in [3, 14]. Many examples of integrable models can be obtained from (3.73) via a *reduction*. This means that one considers cases where the fields depend only on particular combinations of the variables x^i, y^j and the connection 1-forms have special forms (cf [6]). Since reductions do not commute with gauge transformations, it is necessary, however, to consider more generally the bicomplex $(M, \mathsf{D}, \mathcal{D})$, where B is also switched on.

Example. Let the functions depend on x^i, $i = 1, \ldots, n$, only. With $A = 0$ we obtain

$$\mathsf{d}\phi = \partial_i \phi\,\mathsf{d}x^i \qquad \mathcal{D}\phi = B_i \phi\,\mathsf{d}x^i \tag{3.77}$$

and the bicomplex conditions read

$$\partial_i B_j - \partial_j B_i = 0 \qquad B_i B_j - B_j B_i = 0. \tag{3.78}$$

For $m = n$ and assuming that there is a constant metric tensor η_{ij} such that for $B_k = ({B_k}^i{}_j)$ the tensor $B_{ijk} = \eta_{jl} {B_i}^l{}_k$ is totally symmetric, the above equations are equivalent to the WDVV equations [15].

4. Conclusions

By application of gauge transformations and dressings we have constructed bicomplex zero-curvature formulations for several integrable models. The associated linear system then arises in a form with a linear dependence on the (spectral) parameter λ and the existence of an infinite set of conserved densities follows from the general recursive construction of δ-closed elements in the bicomplex associated with the respective model [3]. A relation between our bicomplex formulation and (finite-dimensional) bi-Hamiltonian systems has recently been revealed in [17].

The general iterative construction applies to a much wider range of (weak) bicomplexes than those related to classical soliton equations and integrable models. In particular, generalizations of classical integrable models to corresponding models on noncommutative spaces are obtained in this framework by replacing the ordinary product of functions by the Moyal $*$-product [12] (see also [18] and the references cited there).

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